

Tues
18th Feb 2014

101

Spontaneous Symmetry breaking

Consider real scalar ϕ^4 theory

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi)$$

$$V(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4$$

This theory has a discrete Z_2 symmetry

$$\phi \rightarrow -\phi : \mathcal{L} \rightarrow \mathcal{L}$$

The E.o.M for ϕ

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi = -\frac{\partial V}{\partial \phi} = -[\mu^2 + \lambda \phi^2] \phi$$

The lowest energy classical solution is the

vacuum expectation value

$$\langle 0 | \phi | 0 \rangle \equiv \langle \phi \rangle$$

The Hamiltonian

$$H = \frac{1}{2} \left[\dot{\phi}^2 + (\nabla \phi)^2 + \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]$$

102

tells us the minimum energy state is given

by $\phi = \text{constant}$

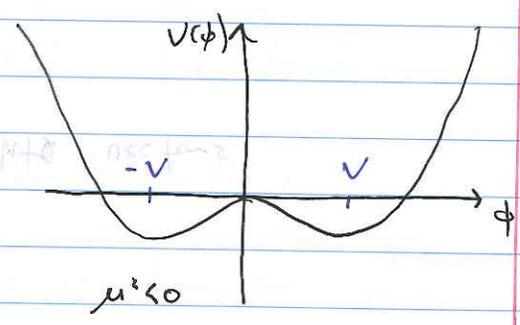
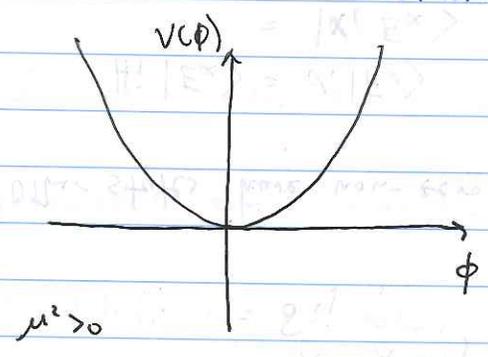
Also, if $\frac{\partial}{\partial x_i} \langle \phi \rangle \neq 0$ or $\frac{\partial}{\partial t} \langle \phi \rangle \neq 0$, the vacuum would violate translation invariance.

So we want $\langle \phi \rangle \quad \frac{\partial V}{\partial \phi} \Big|_{\langle \phi \rangle} = 0$

$\frac{\partial^2 V}{\partial \phi^2} \Big|_{\langle \phi \rangle} > 0$

Vacuum stability $\Rightarrow \lambda \geq 0$

but there is no constraint on μ^2



"tachyons" -

examining theory at "wrong" point - not about true

vacuum

The vacuum of this theory is either

$$\langle \phi \rangle = \pm v = \pm \sqrt{\frac{-\mu^2}{\lambda}}$$

These are two degenerate ground states.

In QM, the g.s. could be a linear combination

$$|g.s.\rangle_{QM} = \frac{1}{\sqrt{2}} (|+v\rangle + |-v\rangle)$$

which would preserve the symmetry. This is

allowed from the finite probability of tunneling

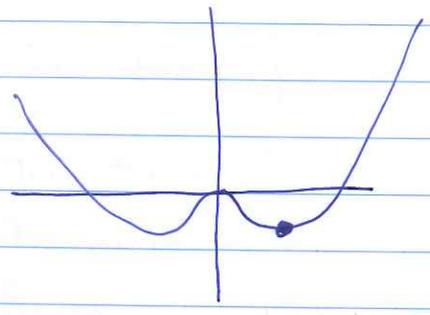
between the two vacuum states.

In an infinite volume, QFT, the barrier potential for tunneling goes to infinity.

Qualitatively: it takes an infinite time for the infinite degrees of freedom to all change $|+v\rangle \rightarrow |-v\rangle$.

Therefore, in ∞ -V QFT, the system will exhibit spontaneous symmetry breaking, picking one or the other vacuum.

Thus, the g.s. does not respect the Z_2 symmetry.



Suppose the vacuum is given by $\langle \phi \rangle = +v$

It is then useful to re-write the \mathcal{L} in terms of new fields, expanded about the vacuum

$$\phi' = \phi - v$$

$$\mathcal{L}(\phi) = \mathcal{L}(\phi' + v)$$

$$= \frac{1}{2} (\partial_\mu \phi')^2 - V(\phi')$$

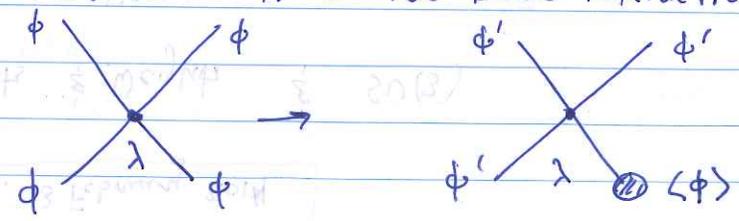
$$V(\phi') = -\frac{\mu^4}{4\lambda} - \mu^2 \phi'^2 + 2v \phi'^3 + \frac{\lambda}{4} \phi'^4$$

The first term is constant \Rightarrow does not effect dynamics but it does contribute to Cosmological Constant.

The second term is a mass

$$m_{\phi'}^2 = -2\mu^2$$

The third term is a new cubic interaction



Notice this terms breaks conservation of $2n$ particle number

The theory for $\mu^2 > 0$ is "boring"

This theory for $\mu^2 < 0$ is difficult to study.
The minimum occurs outside range of validity of theory

$$\langle \phi \rangle = v = \sqrt{\frac{-\mu^2}{\lambda}}, \text{ but for } \lambda \ll 1$$

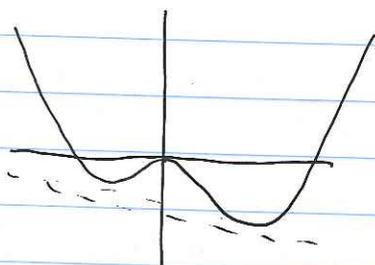
This occurs for "large" values of $\langle \phi \rangle$
as measured by $(-\mu^2)$.

The most interesting case occurs for $\lim_{\mu^2 \rightarrow 0}$

What we will come back to, even for $\mu^2 \rightarrow 0$,
the quantum loop corrections can drive spontaneous
symmetry breaking.

But we can add a small explicit symmetry breaking
term

$$V(\phi) \rightarrow \frac{1}{2} \mu^2 \phi^2 - a \phi + \frac{\lambda}{4} \phi^4$$



This "tadpole" term induces
symmetry breaking

$$\text{For } \mu^2 > 0 \quad \langle \phi \rangle = \frac{a}{\mu^2} + O(a^3)$$

$$\phi': V(\phi') = \frac{1}{2} \mu^2 \phi'^2 + \lambda V \phi'^3 + \frac{1}{4} \lambda \phi'^4$$

For $\mu^2 < 0$

$$v = v_0 + \frac{1}{2} \frac{a}{v_0^2} + O(a^2), \quad v_0 = \sqrt{\frac{-\mu^2}{\lambda}}$$

for sufficiently small a , there is a true minimum, but also a metastable minimum located at

$\langle \phi \rangle_+$

$$V(\phi) = \frac{1}{2} \mu^2 \phi^2 - a\phi + \frac{\lambda}{4} \phi^4$$

$$\frac{\partial V}{\partial \phi} = 0 = \mu^2 \phi - a + \lambda \phi^3$$

$$\lambda V^3 + \mu^2 V - a = 0, \quad V = V_0 + \epsilon$$

$$V_0 = \sqrt{\frac{-\mu^2}{\lambda}}$$

$$\lambda V_0^3 + 3\lambda V_0^2 \epsilon + \mu^2 V_0 + \mu^2 \epsilon - a = 0$$

$$\lambda V_0^3 + \mu^2 V_0 + \epsilon \left[3\lambda V_0^2 + \mu^2 - \frac{a}{\epsilon} \right] = 0$$

$$= 0$$

$$\epsilon = \frac{a}{3\lambda V_0^2 + \mu^2}$$

$$= \frac{a}{2\lambda V_0^2}$$

$$V = -V_0 + \epsilon$$

$$-\lambda V_0^3 + 3\lambda V_0^2 \epsilon - \mu^2 V_0 + \mu^2 \epsilon - a = 0$$

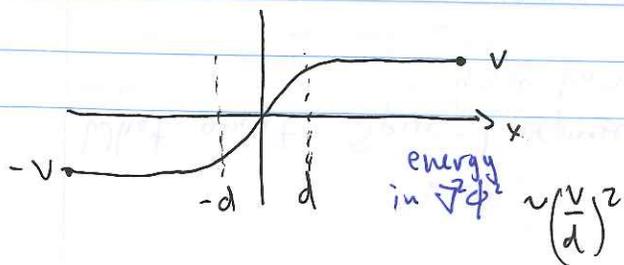
$$-\lambda V_0^3 - \mu^2 V_0 +$$

$$\langle \phi \rangle_+ = V_0 + \frac{a}{2\lambda V_0^2} + O(a^2)$$

$$\langle \phi \rangle_- = -V_0 + \frac{a}{2\lambda V_0^2} + O(a^2)$$

For small a , $\langle \phi \rangle_-$ is metastable.

In inflationary universe, at early times, causally disconnected regions will fall to either $\langle \phi \rangle_{\pm}$. This leads to domain wall solutions



The scale of such theories can be bounded, assuming the energy density of 1 domain wall is less than observed density

What about theories w/ continuous symmetry?

Suppose symmetry G , $|\phi\rangle \xrightarrow{U(G)} U|\phi\rangle$

$\Rightarrow U(G) H_0 U^\dagger(G) = H_0$ hamiltonian is invariant

$$U|\phi_A\rangle = |\phi_B\rangle$$

$$\Rightarrow E_A = E_B$$

$$\begin{aligned}
E_A &= \langle \phi_A | H_0 | \phi_A \rangle \\
&= \langle \phi_A | U^\dagger U H_0 U^\dagger U | \phi_A \rangle \\
&= \langle \phi_B | H_0 | \phi_B \rangle \\
&= E_B
\end{aligned}$$

$$|\phi_A\rangle = \phi_A^\dagger |0\rangle$$

$$|\phi_B\rangle = \phi_B^\dagger |0\rangle$$

$$\Rightarrow U \phi_A U^\dagger = \phi_B$$

But this implies the vacuum is invariant

$$U|0\rangle = |0\rangle$$

If the vacuum is not satisfied for all $U(G)$, then the vacuum spontaneously breaks the symmetry and the degeneracy condition is no longer met.

$$U = e^{i\epsilon Q^a}$$

$U|0\rangle \neq |0\rangle \Rightarrow Q^a|0\rangle \neq 0$ The symmetry charge does not annihilate the vacuum

which is to say there are some states w/ non-vanishing vev.

$$\langle 0 | \phi_j | 0 \rangle \neq 0$$

$$[Q^a, \phi_i] = i t_{ij}^a \phi_j$$

What are consequences of this SSB?

Noether's theorem requires conserved charges for each continuous symmetry

$$\partial_\mu j^\mu(x) = 0 \Rightarrow Q(t) = \int d^3x j_0(x)$$

$$\frac{\partial Q}{\partial t} = 0$$

Consider infinitesimal transformation

$$\phi \rightarrow \phi' = e^{i\epsilon Q} \phi e^{-i\epsilon Q}$$

$$= \phi(x) + i\epsilon [Q, \phi] + \dots$$

Current conservation

$$0 = \int d^3x [\partial_\mu j^\mu(x), \phi(\omega)]$$

$$= \partial_t \int d^3x [j^0(x), \phi(\omega)] + \underbrace{\int d\vec{S} \cdot [\vec{j}(x), \phi(\omega)]}_{0 \text{ for large volume}}$$

$$\Rightarrow \frac{d}{dt} [Q(t), \phi(\omega)] = 0$$

What does this imply for states with non-zero vev?

$$\langle 0 | [Q(t), \phi(\omega)] | 0 \rangle = \eta$$

Insert complete set of states

$$1 = \sum_n \int \frac{d^3p_n}{2p_n^0} \equiv \int \frac{d^3p}{2p^0}$$

$$\begin{aligned} \langle 0 | [Q(t), \phi(x)] | 0 \rangle &= \int d^3x \langle 0 | [j^0(x), \phi(x)] | 0 \rangle \\ &= \int d^3x \left[\langle 0 | j^0(x) | n \rangle \langle n | \phi | 0 \rangle - \langle 0 | \phi | n \rangle \langle n | j^0(x) | 0 \rangle \right] \end{aligned}$$

use translation invariance
 $j^0(x) = e^{i\vec{p}_n \cdot \vec{x}} j^0(0) e^{-i\vec{p}_n \cdot \vec{x}}$

$$\begin{aligned} &= \int d^3x \left[\langle 0 | j^0(0) | n \rangle \langle n | \phi | 0 \rangle e^{-iE_n t} e^{i\vec{p}_n \cdot \vec{x}} - \langle 0 | \phi | n \rangle \langle n | j^0(0) | 0 \rangle e^{+iE_n t} e^{-i\vec{p}_n \cdot \vec{x}} \right] \\ &= \int d^3x (2\pi)^3 \delta^3(\vec{p}_n) \left[\langle 0 | j^0 | n \rangle \langle n | \phi | 0 \rangle e^{-iE_n t} - \langle 0 | \phi | n \rangle \langle n | j^0 | 0 \rangle e^{+iE_n t} \right] \end{aligned}$$

$$2 = \langle 0 | [Q(t), \phi(x)] | 0 \rangle = \int d^3x (2\pi)^3 \delta^3(\vec{p}_n) \left[\langle 0 | j^0 | n \rangle \langle n | \phi | 0 \rangle e^{-iE_n t} - \langle 0 | \phi | n \rangle \langle n | j^0 | 0 \rangle e^{+iE_n t} \right]$$

The left hand side is non-vanishing & time independent

The only way to satisfy this equation is to have a state $|n\rangle$, such that

$$E_n = 0 \quad \text{for} \quad \vec{p}_n = 0$$

For every continuous symmetry which is ~~not~~ invariant for which the vacuum is not invariant, there must be a

- massless
 - spin-0
 - particle
- ↖ $\langle 0 | \phi | 0 \rangle \neq 0$

Goldstone's Theorem

Consider a Complex Scalar field

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi)$$

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

This theory has a continuous $U(1)$ symmetry

$$\phi \rightarrow e^{i\beta} \phi$$

Let us work in the Hermitian basis

$$\phi = \frac{1}{\sqrt{2}} (\sigma + i\pi)$$

$$\mathcal{L} = \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2] - V(\sigma, \pi)$$

$$V(\sigma, \pi) = \frac{1}{2} \mu^2 (\sigma^2 + \pi^2) + \frac{\lambda}{4} (\sigma^2 + \pi^2)^2$$

For $\mu^2 < 0$, the minimum of the potential occurs

$$\sigma^2 + \pi^2 = v^2 = -\frac{\mu^2}{\lambda}$$

The vacuum spontaneously breaks and picks a random direction.

w/out loss of generality, choose the vev

$$\langle \phi \rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

expand \mathcal{L} about $\sigma' = \sigma - v$, $\pi' = \pi$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \sigma')^2 + \frac{1}{2} (\partial_\mu \pi)^2 - V(\sigma', \pi)$$

$$V(\sigma', \pi) = \frac{-\mu^4}{4\lambda} - \mu^2 \sigma'^2 + \lambda v \sigma' (\sigma'^2 + \pi^2) + \frac{\lambda}{4} (\sigma'^2 + \pi^2)^2$$

This describes

massive σ' field
massless π field

$$m_{\sigma'} = \sqrt{2} |\mu|$$

Notice this describes 4 different field couplings with only 2 parameters

The symmetry of \mathcal{L} ensures this is sufficient to renormalize the theory

The conserved current associated w/ $U(1)$ is

$$J_\lambda(x) = -(\partial_\lambda \pi) \sigma' + (\partial_\lambda \sigma') \pi$$

$$Q(t) = \int d^3x J_0(x) = - \int d^3x [(\partial_0 \pi) \sigma' - (\partial_0 \sigma') \pi]$$

$$- [Q, \pi(0)] = -i \sigma(0)$$

$$\langle \pi | \pi(0) | 0 \rangle \neq 0$$

$$- [Q, \sigma(0)] = +i \pi(0)$$

$$\langle 0 | J_0 | \pi \rangle \neq 0$$

Using our earlier eq.

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \delta^3(\vec{p}_n) \left[\langle 0 | J_0 | n \rangle \langle n | \pi | 0 \rangle e^{-iE_n t} - \langle 0 | \pi | n \rangle \langle n | J_0 | 0 \rangle e^{+iE_n t} \right] = +i v$$

$$= \int \frac{d^3\vec{p}}{2p_0} \frac{(2\pi)^3 \delta^3(\vec{p}_\pi)}{(2\pi)^3} \left[\langle 0 | J_0 | \pi(p) \rangle \langle \pi(p) | \pi(0) | 0 \rangle e^{-iE_\pi t} - \langle 0 | \pi(0) | \pi(p) \rangle \langle \pi(p) | J_0 | 0 \rangle e^{+iE_\pi t} \right]$$

take $\langle \pi(p) | \pi(0) | 0 \rangle = 1$

$$\Rightarrow \frac{1}{2p_0} \left[\langle 0 | J_0 | \pi(p) \rangle - \langle \pi(p) | J_0 | 0 \rangle \right] = +i v$$

$$\Rightarrow \langle 0 | J_0 | \pi(p) \rangle = +i v p_0$$

$$v = \langle 0 | \sigma | 0 \rangle$$

Lorentz covariance

$$\langle 0 | J_\mu | \pi(p) \rangle = i v p_\mu$$

$$0 = \langle 0 | \partial_\mu J_\mu | \pi(p) \rangle = v p^2 = v m_\pi^2$$

either $v=0$
or $m_\pi=0$

We will see for QCD, v is related to f_π - pion decay